

# Generalized Taylor formula with integral remainder for Besov-Dunkl spaces

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## Abstract

In the present paper, we propose to prove some properties and estimates of the integral remainder in the generalized Taylor formula associated to the Dunkl operator on the real line and to describe the Besov-Dunkl spaces for which the remainder has a given order.

**Key-words** : Dunkl operator, Dunkl transform, Dunkl translation operators, Dunkl convolution, Besov-Dunkl spaces.

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## 1 Introduction

On the real line, the Dunkl operator is a differential-difference operator introduced in 1989, by C. Dunkl in [6] and is denoted by  $\Lambda_\alpha$  where  $\alpha$  is a real parameter  $> -\frac{1}{2}$ . The operator  $\Lambda_\alpha$  plays a major role in the study of quantum harmonic oscillators governed by Wigner's commutation rules (see [12]). This operator is associated with the reflexion group  $\mathbb{Z}_2$  on  $\mathbb{R}$  and is given by

$$\Lambda_\alpha(f)(x) = \frac{df}{dx}(x) + \frac{2\alpha+1}{x} \left[ \frac{f(x) - f(-x)}{2} \right], \quad f \in \mathcal{C}^1(\mathbb{R}).$$

The Dunkl kernel  $E_\alpha$  related to  $\Lambda_\alpha$  is used to define the Dunkl transform  $\mathcal{F}_\alpha$  which enjoys properties similar to those of the classical Fourier transform. The Dunkl kernel  $E_\alpha$  satisfies a product formula (see [13]). This allows us to define the Dunkl translation  $\tau_x$ ,  $x \in \mathbb{R}$ . As a result, we have the Dunkl convolution  $*_\alpha$  (see next section).

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The classical Taylor formula with integral remainder was extended to the one dimensional Dunkl operator  $\Lambda_\alpha$  in [10]. For  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $a \in \mathbb{R}$ , we have

$$\tau_x(f)(a) = \sum_{p=0}^{k-1} b_p(x) \Lambda_\alpha^p f(a) + R_k(x, f)(a), \quad x \in \mathbb{R} \setminus \{0\},$$

with  $R_k(x, f)(a)$  is the integral remainder of order  $k$  given by

$$R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_{k-1}(x, y) \tau_y(\Lambda_\alpha^k f)(a) A_\alpha(y) dy,$$

where  $\mathcal{E}(\mathbb{R})$  is the space of infinitely differentiable functions on  $\mathbb{R}$  and  $(\Theta_p)_{p \in \mathbb{N}}$ ,  $(b_p)_{p \in \mathbb{N}}$  are two sequences of functions constructed inductively from the function  $A_\alpha$  defined on  $\mathbb{R}$  by  $A_\alpha(x) = |x|^{2\alpha+1}$  (see next section).

Our aim in this paper is to describe the Besov-Dunkl spaces for which the integral remainder in the generalized Taylor formula has a given order.

Let  $0 < \beta < 1$ ,  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$  and  $k$  a positive integer ( $k = 1, 2, \dots$ ). We denote by  $L^p(\mu_\alpha)$  the space of complex-valued functions  $f$ , measurable on  $\mathbb{R}$  such that

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < +\infty,$$

where  $\mu_\alpha$  is a weighted Lebesgue measure associated with the Dunkl operator (see next section). There are many ways to define the Besov spaces (see [7, 11]) and the Besov-Dunkl spaces (see [1, 2, 3]):

- The Besov-Dunkl space of order  $k$  denoted by  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha}$  is the subspace of functions  $f$  in  $\mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$  such that  $\Lambda_\alpha^{k-1}(f) \in L^p(\mu_\alpha)$  and satisfying

$$\int_0^{+\infty} \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty$$

and

$$\sup_{x>0} \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} < +\infty \quad \text{if} \quad q = +\infty,$$

with  $\omega_{p,\alpha}^k(x, f) = \sup_{|y| \leq x} \|R_{k-1}(y, f) - b_{k-1}(y) \Lambda_\alpha^{k-1} f\|_{p,\alpha}$ , where we put for  $k = 1$ ,  $\Lambda_\alpha^0 f = f$  and  $R_0(x, f) = \tau_x(f)$ .

- Put  $\mathcal{D}_{p,\alpha}^k$  the subspace of functions  $f$  in  $\mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$  such that  $\Lambda_\alpha^k(f) \in L^p(\mu_\alpha)$ . We consider the subspace  $\mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha}$  of functions  $f \in \mathcal{D}_{p,\alpha}^{k-1} + \mathcal{D}_{p,\alpha}^k$  satisfying

$$\int_0^{+\infty} \left( \frac{K_{p,\alpha}^k(x, f)}{x^\beta} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty$$

and

$$\sup_{x>0} \frac{K_{p,\alpha}^k(x, f)}{x^\beta} < +\infty \quad \text{if} \quad q = +\infty,$$

where  $K_{p,\alpha}^k$  is the Peetre K-functional given by

$$K_{p,\alpha}^k(x, f) = \inf_{f=f_0+f_1} \left\{ \|\Lambda_\alpha^{k-1}(f_0)\|_{p,\alpha} + x \|\Lambda_\alpha^k(f_1)\|_{p,\alpha}, f_0 \in \mathcal{D}_{p,\alpha}^{k-1}, f_1 \in \mathcal{D}_{p,\alpha}^k \right\}.$$

- $\tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha}$  denote the subspace of functions  $f$  in  $\mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$  such that  $\Lambda_\alpha^{k-1}(f) \in L^p(\mu_\alpha)$  and satisfying

$$\int_0^{+\infty} \left( \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} < +\infty \quad \text{if} \quad q < +\infty$$

and

$$\sup_{x>0} \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} < +\infty \quad \text{if} \quad q = +\infty,$$

with  $\tilde{\omega}_{p,\alpha}^k(x, f) = \|R_{k-1}(x, f) + R_{k-1}(-x, f) - (b_{k-1}(x) + b_{k-1}(-x))\Lambda_\alpha^{k-1}f\|_{p,\alpha}$ , where we put for  $k = 1$ ,  $\Lambda_\alpha^0 f = f$ ,  $R_0(x, f) = \tau_x(f)$  and  $R_0(-x, f) = \tau_{-x}(f)$ .

- Let  $\phi \in \mathcal{S}_*(\mathbb{R})$  such that  $\int_0^{+\infty} x^{2i} \phi(x) d\mu_\alpha(x) = 0$ , for all  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$  where  $\mathcal{S}_*(\mathbb{R})$  is the space of even Schwartz functions on  $\mathbb{R}$  (see Example 4.2, section 4). We shall denote by  $\mathcal{C}_{p,q}^{k,\beta,\alpha}$  the subspace of functions  $f$  in  $\mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{2i}(f) \in L^p(\mu_\alpha)$ ,  $0 \leq i \leq [\frac{k-1}{2}]$  and satisfying

$$\int_0^{+\infty} \left( \frac{\|f *_\alpha \phi_t\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} < +\infty \quad \text{if} \quad q < +\infty$$

and

$$\sup_{t>0} \frac{\|f *_\alpha \phi_t\|_{p,\alpha}}{t^{\beta+k-1}} < +\infty \quad \text{if} \quad q = +\infty,$$

where  $\phi_t$  is the dilation of  $\phi$  given by  $\phi_t(x) = \frac{1}{t^{2(\alpha+1)}} \phi(\frac{x}{t})$ , for all  $t \in (0, +\infty)$  and  $x \in \mathbb{R}$ .

In this paper, we give some properties and estimates of the integral remainder of order  $k$  and we establish that

$$\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha} \quad ; \quad \tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{C}_{p,q}^{k,\beta,\alpha}.$$

Note that we have  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \subset \tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha}$  (see section 4).

The results obtained in this paper are an extension to the Dunkl theory on the real line of those obtained in [5, 8]. More precisely, in [5], the authors showed in the classical case and for  $k = 1$  that  $\tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{C}_{p,q}^{k,\beta,\alpha}$ .

The contents of this paper are as follows.

In section 2, we collect some basic definitions and results about harmonic analysis associated with the Dunkl operator  $\Lambda_\alpha$ .

In section 3, we prove some properties and estimates of the integral remainder of order  $k$ .

Finally, we establish in the section 4, the coincidence between the different characterizations of the Besov-Dunkl spaces.

Along this paper, we use  $c$  to represent a suitable positive constant which is not necessarily the same in each occurrence.

## 2 Preliminaries

In this section, we recall some notations and results in Dunkl theory on  $\mathbb{R}$  and we refer for more details to [4, 6, 13].

For  $\lambda \in \mathbb{C}$ , the initial problem

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},$$

has a unique solution  $E_\alpha(\lambda)$  called Dunkl kernel given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha + 1)} j_{\alpha+1}(i\lambda x), \quad x \in \mathbb{R},$$

where  $j_\alpha$  is the normalized Bessel function of the first kind and order  $\alpha$ , defined by

$$j_\alpha(\lambda x) = \begin{cases} 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(\lambda x)}{(\lambda x)^\alpha} & \text{if } \lambda x \neq 0 \\ 1 & \text{if } \lambda x = 0, \end{cases}$$

here  $J_\alpha$  is the Bessel function of first kind and order  $\alpha$ .

We have for all  $x \in \mathbb{R}$ , the function  $\lambda \rightarrow j_\alpha(\lambda x)$  is even on  $\mathbb{R}$  and

$$|E_\alpha(-i\lambda x)| \leq 1.$$

Let  $A_\alpha$  the function defined on  $\mathbb{R}$  by

$$A_\alpha(x) = |x|^{2\alpha+1}, \quad x \in \mathbb{R},$$

and  $\mu_\alpha$  the weighted Lebesgue measure on  $\mathbb{R}$  given by

$$d\mu_\alpha(x) = \frac{A_\alpha(x)}{2^{\alpha+1}\Gamma(\alpha + 1)} dx. \quad (2.1)$$

For every  $1 \leq p \leq +\infty$ , we denote by  $L^p(\mu_\alpha)$  the space of complex-valued functions  $f$ , measurable on  $\mathbb{R}$  such that

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < +\infty, \quad \text{if } p < +\infty,$$

and

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)| < +\infty.$$

There exists an analogue of the classical Fourier transform with respect to the Dunkl kernel called the Dunkl transform and denoted by  $\mathcal{F}_\alpha$ . The Dunkl transform enjoys properties similar to those of the classical Fourier transform and is defined for  $f \in L^1(\mu_\alpha)$  by

$$\mathcal{F}_\alpha(f)(x) = \int_{\mathbb{R}} f(y) E_\alpha(-ixy) d\mu_\alpha(y), \quad x \in \mathbb{R}.$$

For all  $x, y, z \in \mathbb{R}$ , we consider

$$W_\alpha(x, y, z) = \frac{(\Gamma(\alpha + 1)^2)}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} (1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x}) \Delta_\alpha(x, y, z)$$

where

$$b_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in \mathbb{R} \setminus \{0\}, z \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_\alpha(x, y, z) = \begin{cases} \frac{((|x|+|y|)^2 - z^2)[z^2 - (|x|-|y|)^2])^{\alpha-\frac{1}{2}}}{|xyz|^{2\alpha}} & \text{if } |z| \in S_{x,y} \\ 0 & \text{otherwise} \end{cases}$$

where

$$S_{x,y} = [|x| - |y|, |x| + |y|].$$

The kernel  $W_\alpha$ , is even and we have

$$W_\alpha(x, y, z) = W_\alpha(y, x, z) = W_\alpha(-x, z, y) = W_\alpha(-z, y, -x)$$

and

$$\int_{\mathbb{R}} |W_\alpha(x, y, z)| d\mu_\alpha(z) \leq \sqrt{2}.$$

The Dunkl kernel  $E_\alpha$  satisfies the following product formula

$$E_\alpha(ixt)E_\alpha(iyt) = \int_{\mathbb{R}} E_\alpha(itz) d\gamma_{x,y}(z), \quad x, y, t \in \mathbb{R},$$

where  $\gamma_{x,y}$  is a signed measure on  $\mathbb{R}$  given by

$$d\gamma_{x,y}(z) = \begin{cases} W_\alpha(x, y, z) d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R} \setminus \{0\} \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \quad (2.2)$$

with  $\text{supp} \gamma_{x,y} = S_{x,y} \cup (-S_{x,y})$ .

For  $x, y \in \mathbb{R}$  and  $f$  a continuous function on  $\mathbb{R}$ , the Dunkl translation operator  $\tau_x$  given by

$$\tau_x(f)(y) = \int_{\mathbb{R}} f(z) d\gamma_{x,y}(z)$$

satisfies the following properties :

- $\tau_x$  is a continuous linear operator from  $\mathcal{E}(\mathbb{R})$  into itself.
- For all  $f \in \mathcal{E}(\mathbb{R})$ , we have

$$\tau_x(f)(y) = \tau_y(f)(x) \quad \text{and} \quad \tau_0(f)(x) = f(x) \quad (2.3)$$

$$\tau_x \circ \tau_y = \tau_y \circ \tau_x \quad \text{and} \quad \Lambda_\alpha \circ \tau_x = \tau_x \circ \Lambda_\alpha. \quad (2.4)$$

- For all  $x \in \mathbb{R}$ , the operator  $\tau_x$  extends to  $L^p(\mu_\alpha)$ ,  $p \geq 1$  and we have for  $f \in L^p(\mu_\alpha)$

$$\|\tau_x(f)\|_{p,\alpha} \leq \sqrt{2} \|f\|_{p,\alpha}. \quad (2.5)$$

The Dunkl convolution  $f *_{\alpha} g$  of two continuous functions  $f$  and  $g$  on  $\mathbb{R}$  with compact support, is defined by

$$(f *_{\alpha} g)(x) = \int_{\mathbb{R}} \tau_x(f)(-y)g(y)d\mu_{\alpha}(y), \quad x \in \mathbb{R}.$$

The convolution  $*_{\alpha}$  is associative and commutative and satisfies the following property:

- Assume that  $p, q, r \in [1, +\infty[$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$  (the Young condition). Then the map  $(f, g) \rightarrow f *_{\alpha} g$  defined on  $C_c(\mathbb{R}) \times C_c(\mathbb{R})$ , extends to a continuous map from  $L^p(\mu_{\alpha}) \times L^q(\mu_{\alpha})$  to  $L^r(\mu_{\alpha})$  and we have

$$\|f *_{\alpha} g\|_{r,\alpha} \leq \sqrt{2}\|f\|_{p,\alpha}\|g\|_{q,\alpha}. \quad (2.6)$$

- For all  $f \in L^1(\mu_{\alpha})$ ,  $g \in L^2(\mu_{\alpha})$  and  $h \in L^p(\mu_{\alpha})$ ,  $1 \leq p < +\infty$ , we have

$$\mathcal{F}_{\alpha}(f *_{\alpha} g) = \mathcal{F}_{\alpha}(f)\mathcal{F}_{\alpha}(g) \quad \text{and} \quad \tau_t(f *_{\alpha} h) = \tau_t(f) *_{\alpha} h = f *_{\alpha} \tau_t(h), \quad t \in \mathbb{R}. \quad (2.7)$$

It has been shown in [10], the following generalized Taylor formula with integral remainder:

**Proposition 2.1** *For  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $a \in \mathbb{R}$ , we have*

$$\tau_x f(a) = \sum_{p=0}^{k-1} b_p(x) \Lambda_{\alpha}^p f(a) + R_k(x, f)(a), \quad x \in \mathbb{R} \setminus \{0\}, \quad (2.8)$$

with  $R_k(x, f)(a)$  is the integral remainder of order  $k$  given by

$$R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_{k-1}(x, y) \tau_y(\Lambda_{\alpha}^k f)(a) A_{\alpha}(y) dy, \quad (2.9)$$

where

$$i) \quad b_{2m}(x) = \frac{1}{(\alpha+1)_m m!} \left(\frac{x}{2}\right)^{2m}, \quad b_{2m+1}(x) = \frac{1}{(\alpha+1)_{m+1} m!} \left(\frac{x}{2}\right)^{2m+1}, \quad \text{for all } m \in \mathbb{N}.$$

$$ii) \quad \Theta_{k-1}(x, y) = u_{k-1}(x, y) + v_{k-1}(x, y) \quad \text{with} \quad u_0(x, y) = \frac{\text{sgn}(x)}{2A_{\alpha}(x)}, \quad v_0(x, y) = \frac{\text{sgn}(y)}{2A_{\alpha}(y)},$$

$$\text{and} \quad u_k(x, y) = \int_{|y|}^{|x|} v_{k-1}(x, z) dz, \quad v_k(x, y) = \frac{\text{sgn}(y)}{A_{\alpha}(y)} \int_{|y|}^{|x|} u_{k-1}(x, z) A_{\alpha}(z) dz.$$

According to ([15], Lemma 2.2), the Dunkl operator  $\Lambda_{\alpha}$  have the following regularity properties:

$$\Lambda_{\alpha} \text{ leaves } \mathcal{C}_c^{\infty}(\mathbb{R}) \text{ and the Schwartz space } \mathcal{S}(\mathbb{R}) \text{ invariant.} \quad (2.10)$$

### 3 Some properties of the integral remainder of order $k$

In this section, we prove some properties and estimates of the integral remainder in the generalized Taylor formula.

**Remark 3.1** Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $x \in \mathbb{R} \setminus \{0\}$ .

1. From Proposition 2.1, we have

$$\begin{aligned} R_k(x, f) &= \tau_x(f) - f - b_1(x)\Lambda_\alpha f \dots - b_{k-1}(x)\Lambda_\alpha^{k-1}f \\ &= R_{k-1}(x, f) - b_{k-1}(x)\Lambda_\alpha^{k-1}f, \end{aligned} \quad (3.1)$$

where we put for  $k = 1$ ,  $R_0(x, f) = \tau_x(f)$ . Observe that

$$R_1(x, f) = R_0(x, f) - b_0(x)\Lambda_\alpha^0 f = \tau_x(f) - f.$$

2. According to ([10], p.352) and Proposition 2.1, i), we have

$$\begin{aligned} \int_{-|x|}^{|x|} |\Theta_{k-1}(x, y)| A_\alpha(y) dy &\leq b_k(|x|) + |x| b_{k-1}(|x|) \\ &\leq c |x|^k. \end{aligned} \quad (3.2)$$

3. Note that the function  $y \mapsto \tau_y(f) - f$  is continuous on  $\mathbb{R}$  (see [9], Lemma 1, (ii)), which implies that the same is true for the function  $y \mapsto R_k(y, f)$ .

**Lemma 3.1** Let  $k = 1, 2, \dots$ , and  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1}f \in L^p(\mu_\alpha)$ . Then we have

$$\|R_{k-1}(x, f)\|_{p, \alpha} \leq c |x|^{k-1} \|\Lambda_\alpha^{k-1}f\|_{p, \alpha}, \quad x \in \mathbb{R} \setminus \{0\}. \quad (3.3)$$

**Proof.** Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1}f \in L^p(\mu_\alpha)$  and  $x \in \mathbb{R} \setminus \{0\}$ . For  $k = 1$ , by (2.5), it's clear that  $\|R_0(x, f) = \tau_x(f)\|_{p, \alpha} \leq c \|f\|_{p, \alpha}$ . Using the Minkowski's inequality for integrals, (2.5) and (2.9), we have for  $k \geq 2$

$$\begin{aligned} \|R_{k-1}(x, f)\|_{p, \alpha} &\leq \int_{-|x|}^{|x|} |\Theta_{k-2}(x, y)| \|\tau_y(\Lambda_\alpha^{k-1}f)\|_{p, \alpha} A_\alpha(y) dy \\ &\leq c \|\Lambda_\alpha^{k-1}f\|_{p, \alpha} \int_{-|x|}^{|x|} |\Theta_{k-2}(x, y)| A_\alpha(y) dy. \end{aligned}$$

Using (3.2), we deduce our result. ■

**Remark 3.2** Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1}f \in L^p(\mu_\alpha)$  and  $x \in \mathbb{R} \setminus \{0\}$ . Then we have by (3.1), (3.3) and Proposition 2.1, i),

$$\begin{aligned} \|R_k(x, f)\|_{p, \alpha} &= \|R_{k-1}(x, f) + b_{k-1}(x)\Lambda_\alpha^{k-1}f\|_{p, \alpha} \\ &\leq \|R_{k-1}(x, f)\|_{p, \alpha} + \|b_{k-1}(x)\Lambda_\alpha^{k-1}f\|_{p, \alpha} \\ &\leq c |x|^{k-1} \|\Lambda_\alpha^{k-1}f\|_{p, \alpha}. \end{aligned} \quad (3.4)$$

**Lemma 3.2** For  $x \in \mathbb{R} \setminus \{0\}$  and  $p \in \mathbb{N}$ , we have

$$\int_{-|x|}^{|x|} \Theta_0(x, y) b_p(y) A_\alpha(y) dy = b_{p+1}(y). \quad (3.5)$$

**Proof.** Let  $x \in \mathbb{R} \setminus \{0\}$ . Using Proposition 2.1, we have

- If  $p = 2m$ ,  $m \in \mathbb{N}$ ,

$$\begin{aligned} \int_{-|x|}^{|x|} \Theta_0(x, y) b_{2m}(y) A_\alpha(y) dy &= \int_{-|x|}^{|x|} u_0(x, y) b_{2m}(y) A_\alpha(y) dy + \int_{-|x|}^{|x|} v_0(x, y) b_{2m}(y) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \frac{\operatorname{sgn}(x) |y|^{2\alpha+1}}{2|x|^{2\alpha+1}} b_{2m}(y) dy + \int_{-|x|}^{|x|} \frac{\operatorname{sgn}(y)}{2} b_{2m}(y) dy \\ &= \frac{x}{2^{2m} |x|^{2\alpha+2} (\alpha+1)_m m!} \int_0^{|x|} y^{2\alpha+2m+1} dy \\ &= \frac{x}{2^{2m} (\alpha+1)_m m!} \frac{|x|^{2m}}{2(\alpha+m+1)} \\ &= b_{2m+1}(x). \end{aligned}$$

- If  $p = 2m + 1$ ,  $m \in \mathbb{N}$ , we get

$$\begin{aligned} \int_{-|x|}^{|x|} \Theta_0(x, y) b_{2m+1}(y) A_\alpha(y) dy &= \int_{-|x|}^{|x|} u_0(x, y) b_{2m+1}(y) A_\alpha(y) dy + \int_{-|x|}^{|x|} v_0(x, y) b_{2m+1}(y) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \frac{\operatorname{sgn}(x) |y|^{2\alpha+1}}{2|x|^{2\alpha+1}} b_{2m+1}(y) dy + \int_{-|x|}^{|x|} \frac{\operatorname{sgn}(y)}{2} b_{2m+1}(y) dy \\ &= \frac{1}{2^{2m+1} (\alpha+1)_{m+1} m!} \int_0^{|x|} y^{2m+1} dy \\ &= \frac{1}{2^{2m+1} (\alpha+1)_{m+1} m!} \frac{|x|^{2m+2}}{2(m+1)} \\ &= b_{2m+2}(x). \end{aligned}$$

Our Lemma is proved. ■

**Lemma 3.3** Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$ ,  $x \in \mathbb{R} \setminus \{0\}$  and  $a \in \mathbb{R}$ . Then we have,

$$R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) R_{k-1}(y, \Lambda_\alpha f)(a) A_\alpha(y) dy. \quad (3.6)$$

**Proof.** Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$ ,  $x \in \mathbb{R} \setminus \{0\}$  and  $a \in \mathbb{R}$ .



- We have from (2.8), (2.9) and the fact that  $R_0(y, \Lambda_\alpha f)(a) = \tau_y(\Lambda_\alpha f)$

$$R_1(x, f)(a) = (\tau_x(f) - f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) \tau_y(\Lambda_\alpha f)(a) A_\alpha(y) dy,$$

hence the property (3.6) is true for  $k = 1$ .

- Suppose that

$$R_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) R_{k-1}(y, \Lambda_\alpha f)(a) A_\alpha(y) dy,$$

then by (3.1) and (3.5) again, we get

$$\begin{aligned} \int_{-|x|}^{|x|} \Theta_0(x, y) R_k(y, \Lambda_\alpha f)(a) A_\alpha(y) dy &= \int_{-|x|}^{|x|} \Theta_0(x, y) [R_{k-1}(y, \Lambda_\alpha f) - b_{k-1}(y) \Lambda_\alpha^k f](a) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \Theta_0(x, y) R_{k-1}(y, \Lambda_\alpha f)(a) A_\alpha(y) dy \\ &\quad - \int_{-|x|}^{|x|} \Theta_0(x, y) b_{k-1}(y) \Lambda_\alpha^k f(a) A_\alpha(y) dy \\ &= R_k(x, f)(a) - \Lambda_\alpha^k f(a) \int_{-|x|}^{|x|} \Theta_0(x, y) b_{k-1}(y) A_\alpha(y) dy \\ &= R_k(x, f)(a) - b_k(x) \Lambda_\alpha^k f(a) \\ &= R_{k+1}(x, f)(a). \end{aligned}$$

Hence by induction, we deduce our result. ■

**Lemma 3.4** *Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$ ,  $x \in \mathbb{R} \setminus \{0\}$  and  $a \in \mathbb{R}$ . We denote by*

$$I_1(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) \tau_y(f)(a) A_\alpha(y) dy,$$

and for  $k \geq 2$

$$I_k(x, f)(a) = \int_{-|x|}^{|x|} \Theta_0(x, y) I_{k-1}(y, f)(a) A_\alpha(y) dy.$$

Then, we have

$$\Lambda_\alpha^{k+1}(I_k(x, f))(a) = \Lambda_\alpha^k(I_k(x, \Lambda_\alpha f))(a), \quad (3.7)$$

$$\text{and} \quad \Lambda_\alpha^k I_k(x, f)(a) = R_k(x, f)(a). \quad (3.8)$$

**Proof.** Let  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$ ,  $x \in \mathbb{R} \setminus \{0\}$  and  $a \in \mathbb{R}$ .

- Using (2.4), we have

$$\begin{aligned}\Lambda_\alpha^2(I_1(x, f))(a) &= \int_{-|x|}^{|x|} \Theta_0(x, y) \Lambda_\alpha \tau_y(\Lambda_\alpha f)(a) A_\alpha(y) dy \\ &= \Lambda_\alpha(I_1(x, \Lambda_\alpha f))(a).\end{aligned}$$

Suppose that

$$\Lambda_\alpha^{k+1}(I_k(x, f))(a) = \Lambda_\alpha^k(I_k(x, \Lambda_\alpha f))(a),$$

then we have

$$\begin{aligned}\Lambda_\alpha^{k+2}(I_{k+1}(x, f))(a) &= \int_{-|x|}^{|x|} \Theta_0(x, y) \Lambda_\alpha(\Lambda_\alpha^{k+1} I_k(y, f))(a) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \Theta_0(x, y) \Lambda_\alpha(\Lambda_\alpha^k I_k(y, \Lambda_\alpha f))(a) A_\alpha(y) dy \\ &= \Lambda_\alpha^{k+1}(I_{k+1}(x, \Lambda_\alpha f))(a),\end{aligned}$$

hence by induction, we obtain our result.

- From (2.4), (2.9) and (3.6), we can write

$$\begin{aligned}\Lambda_\alpha(I_1(x, f))(a) &= \int_{-|x|}^{|x|} \Theta_0(x, y) \Lambda_\alpha(\tau_y f)(a) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \Theta_0(x, y) \tau_y(\Lambda_\alpha f)(a) A_\alpha(y) dy \\ &= R_1(x, f)(a).\end{aligned}$$

Suppose that

$$\Lambda_\alpha^k(I_k(x, f))(a) = R_k(x, f)(a),$$

then by (3.6) and (3.7), we have

$$\begin{aligned}\Lambda_\alpha^{k+1}(I_{k+1}(x, f))(a) &= \int_{-|x|}^{|x|} \Theta_0(x, y) \Lambda_\alpha^{k+1}(I_k(y, f))(a) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \Theta_0(x, y) \Lambda_\alpha^k(I_k(y, \Lambda_\alpha f))(a) A_\alpha(y) dy \\ &= \int_{-|x|}^{|x|} \Theta_0(x, y) R_k(y, \Lambda_\alpha f)(a) A_\alpha(y) dy \\ &= R_{k+1}(x, f)(a).\end{aligned}$$

By induction, we deduce our result. ■

**Remark 3.3** For  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $x \in \mathbb{R} \setminus \{0\}$ , we observe from Proposition 2.1 that

$$\begin{aligned} R_k(x, f) + R_k(-x, f) &= \tau_x(f) + \tau_{-x}(f) - \sum_{p=0}^{k-1} (b_p(x) + b_p(-x)) \Lambda_\alpha^p f \\ &= \tau_x(f) + \tau_{-x}(f) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}(x) \Lambda_\alpha^{2i} f. \end{aligned} \quad (3.9)$$

## 4 Characterizations of Besov-Dunkl spaces of order $k$

In this section, we establish respectively that  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha}$  and  $\tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{C}_{p,q}^{k,\beta,\alpha}$ . Before proving these results, we begin with a useful remarks, a proposition containing sufficient conditions and an example.

**Remark 4.1** For  $k = 1, 2, \dots$ ,  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1}(f) \in L^p(\mu_\alpha)$  and  $x \in (0, +\infty)$ , we can assert from (3.1) and (3.4) that

$$1/ \omega_{p,\alpha}^k(x, f) = \sup_{|y| \leq x} \|R_k(y, f)\|_{p,\alpha}.$$

$$2/ \tilde{\omega}_{p,\alpha}^k(x, f) = \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha}.$$

**Proposition 4.1** Let  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$ ,  $0 < \beta < 1$ ,  $f \in \mathcal{E}(\mathbb{R})$  and  $k = 1, 2, \dots$ . If  $\Lambda_\alpha^{k-1}(f)$  and  $\Lambda_\alpha^k(f)$  are in  $L^p(\mu_\alpha)$ , then  $f \in \mathcal{B}^k \mathcal{D}_{\beta,\alpha}^{p,q}$ .

**Proof.** Let  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$ ,  $0 < \beta < 1$  and  $f \in \mathcal{E}(\mathbb{R})$  such that  $\Lambda_\alpha^{k-1}(f)$ ,  $\Lambda_\alpha^k(f)$  are in  $L^p(\mu_\alpha)$  for  $k = 1, 2, \dots$ . By (3.3) and (3.4), we obtain for  $x \in (0, +\infty)$

$$\omega_{p,\alpha}^k(x, f) \leq c x^k \|\Lambda_\alpha^k f\|_{p,\alpha} \quad \text{and} \quad \omega_{p,\alpha}^k(x, f) \leq c x^{k-1} \|\Lambda_\alpha^{k-1} f\|_{p,\alpha}.$$

Then we can write,

$$\int_0^{+\infty} \left( \frac{\omega_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} \leq c \int_0^1 \left( \frac{\|\Lambda_\alpha^k f\|_{p,\alpha}}{x^{\beta-1}} \right)^q \frac{dx}{x} + c \int_1^{+\infty} \left( \frac{\|\Lambda_\alpha^{k-1} f\|_{p,\alpha}}{x^\beta} \right)^q \frac{dx}{x} < +\infty.$$

Here when  $q = +\infty$ , we make the usual modification. ■

**Example 4.1** From (2.10) and Proposition 4.1, we can assert that the spaces  $\mathcal{C}_c^\infty(\mathbb{R})$  and  $\mathcal{S}(\mathbb{R})$  are included in  $\mathcal{B}^k \mathcal{D}_{\beta,\alpha}^{p,q}$ .

**Remark 4.2** By the fact that  $\tilde{\omega}_{p,\alpha}^k(x, f) \leq 2 \omega_{p,\alpha}^k(x, f)$ , we have clearly  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \subset \tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ . Observe that for  $k = 1$ , we have

$$\omega_{p,\alpha}^k(x, f) = \sup_{|y| \leq x} \|\tau_y(f) - f\|_{p,\alpha} \quad \text{and} \quad \tilde{\omega}_{p,\alpha}^k(x, f) = \|\tau_x(f) + \tau_{-x}(f) - 2f\|_{p,\alpha}.$$

**Theorem 4.1** *Let  $0 < \beta < 1$ ,  $k = 1, 2, \dots$ ,  $1 \leq p < +\infty$  and  $1 \leq q \leq +\infty$ , then*

$$\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha}.$$

**Proof.** We start with the proof of the inclusion  $\mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha} \subset \mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ . For  $f \in \mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ ,  $f = f_0 + f_1$ ,  $f_0 \in \mathcal{D}_{p,\alpha}^{k-1}$  and  $f_1 \in \mathcal{D}_{p,\alpha}^k$ , we have by (3.3)

$$\begin{aligned} \omega_{p,\alpha}^k(x, f_1) &= \sup_{|y| \leq x} \|R_k(y, f_1)\|_{p,\alpha} \\ &\leq c x^k \|\Lambda_\alpha^k f_1\|_{p,\alpha}, \quad x \in (0, +\infty). \end{aligned} \quad (4.1)$$

Using (3.4), we obtain

$$\begin{aligned} \omega_{p,\alpha}^k(x, f_0) &\leq \sup_{|y| \leq x} \|R_{k-1}(y, f_0)\|_{p,\alpha} + \sup_{|y| \leq x} \|b_{k-1}(y) \Lambda_\alpha^{k-1} f_0\|_{p,\alpha} \\ &\leq c x^{k-1} \|\Lambda_\alpha^{k-1} f_0\|_{p,\alpha}, \quad x \in (0, +\infty). \end{aligned} \quad (4.2)$$

Hence by (4.1) et (4.2), we deduce that

$$\omega_{p,\alpha}^k(x, f) \leq c x^{k-1} K_{p,\alpha}^k(x, f),$$

then,  $f \in \mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ .

Let prove now the inclusion  $\mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha} \subset \mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ . For  $f \in \mathcal{B}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ , we take for  $x \in (0, +\infty)$

$$f_1 = \frac{1}{b_k(x)} I_k(x, f).$$

Using (3.8), we obtain

$$\begin{aligned} x \|\Lambda_\alpha^k f_1\|_{p,\alpha} &\leq x |b_k(x)|^{-1} \omega_{p,\alpha}^k(x, f) \\ &\leq c \frac{\omega_{p,\alpha}^k(x, f)}{x^{k-1}}. \end{aligned} \quad (4.3)$$

On the other hand, put  $f_0 = f - f_1$ , we can write using (3.5)

$$f_0 = -\frac{1}{b_k(x)} \int_{-x}^x \Theta_0(x, y) (I_{k-1}(y, f) - b_{k-1}(y) f) A_\alpha(y) dy.$$

From (3.1) and (3.8), we obtain

$$\Lambda_\alpha^{k-1} f_0 = -\frac{1}{b_k(x)} \int_{-x}^x \Theta_0(x, y) R_k(y, f) A_\alpha(y) dy.$$

By Minkowski's inequality for integrals and (3.2), we get

$$\begin{aligned} \|\Lambda_\alpha^{k-1} f_0\|_{p,\alpha} &\leq |b_k(x)|^{-1} \int_{-x}^x |\Theta_0(x, y)| \|R_k(y, f)\|_{p,\alpha} A_\alpha(y) dy \\ &\leq c x^{-k} \omega_{p,\alpha}^k(x, f) \int_{-x}^x |\Theta_0(x, y)| A_\alpha(y) dy \\ &\leq c \frac{\omega_{p,\alpha}^k(x, f)}{x^{k-1}}. \end{aligned} \quad (4.4)$$

By (4.3) et (4.4), we deduce that

$$K_{p,\alpha}^k(x, f) \leq c \frac{\omega_{p,\alpha}^k(x, f)}{x^{k-1}},$$

then,  $f \in \mathcal{K}^k \mathcal{D}_{p,q}^{\beta,\alpha}$ . Our theorem is proved.  $\blacksquare$

In order to establish that  $\tilde{\mathcal{B}}^k \mathcal{D}_{p,q}^{\beta,\alpha} = \mathcal{C}_{p,q}^{k,\beta,\alpha}$ , we need to prove some useful lemmas.

**Lemma 4.1** *Let  $k = 1, 2, \dots$ ,  $1 \leq p < +\infty$ ,  $\phi \in \mathcal{S}_*(\mathbb{R})$  such that  $\int_0^{+\infty} x^{2i} \phi(x) d\mu_\alpha(x) = 0$ , for all  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$  and  $r > 0$ , then there exists a constant  $c > 0$  such that for all  $f \in \mathcal{E}(\mathbb{R}) \cap L^p(\mu_\alpha)$  satisfying  $\Lambda_\alpha^{k-1} f \in L^p(\mu_\alpha)$  and  $t > 0$ , we have*

$$\|\phi_t *_\alpha f\|_{p,\alpha} \leq c \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{2(\alpha+1)}, \left( \frac{t}{x} \right)^r \right\} \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} \frac{dx}{x}. \quad (4.5)$$

**Proof.** Let  $t > 0$ , we have for  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$ ,

$$\int_0^{+\infty} x^{2i} \phi(x) d\mu_\alpha(x) = 0 \implies \int_0^{+\infty} x^{2i} \phi_t(x) d\mu_\alpha(x) = 0, \quad (4.6)$$

and

$$\begin{aligned} (\phi_t *_\alpha f)(y) &= \int_{\mathbb{R}} \phi_t(x) \tau_y(f)(-x) d\mu_\alpha(x) \\ &= \int_{\mathbb{R}} \phi_t(x) \tau_y(f)(x) d\mu_\alpha(x). \end{aligned}$$

Then using (2.3), (3.9), (4.6) and Proposition 2.1, we can write for  $y \in \mathbb{R}$

$$\begin{aligned} 2(\phi_t *_\alpha f)(y) &= \int_{\mathbb{R}} \phi_t(x) \left( \tau_y(f)(x) + \tau_y(f)(-x) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_\alpha^{2i} f(y) \right) d\mu_\alpha(x) \\ &= 2 \int_0^{+\infty} \phi_t(x) \left( \tau_x(f)(y) + \tau_{-x}(f)(y) - 2 \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i}(x) \Lambda_\alpha^{2i} f(y) \right) d\mu_\alpha(x) \\ &= 2 \int_0^{+\infty} \phi_t(x) (R_k(x, f)(y) + R_k(-x, f)(y)) d\mu_\alpha(x). \end{aligned}$$

By Minkowski's inequality for integrals, we obtain

$$\begin{aligned} \|\phi_t *_\alpha f\|_{p,\alpha} &\leq \int_0^{+\infty} |\phi_t(x)| \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} d\mu_\alpha(x) \\ &\leq c \int_0^{+\infty} \left( \frac{x}{t} \right)^{2(\alpha+1)} \left| \phi \left( \frac{x}{t} \right) \right| \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} \frac{dx}{x} \quad (4.7) \end{aligned}$$

$$\leq c \int_0^{+\infty} \left( \frac{x}{t} \right)^{2(\alpha+1)} \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} \frac{dx}{x}. \quad (4.8)$$

On the other hand, since  $\phi \in \mathcal{S}_*(\mathbb{R})$ , then from (4.7) and for  $r > 0$  there exists a constant  $c$  such that

$$\|\phi_t *_{\alpha} f\|_{p,\alpha} \leq c \int_0^{+\infty} \left(\frac{t}{x}\right)^r \|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} \frac{dx}{x}. \quad (4.9)$$

Using (4.8) and (4.9), we deduce our result.  $\blacksquare$

**Example 4.2** According to ([14], Example 3.3, (2)), the generalized Hermite polynomials on  $\mathbb{R}$ , denoted by  $H_n^{\alpha+\frac{1}{2}}$ ,  $n \in \mathbb{N}$  are orthogonal with respect to the measure  $e^{-x^2} d\mu_{\alpha}(x)$  and can be written as

$$H_{2n}^{\alpha+\frac{1}{2}}(x) = (-1)^n 2^{2n} n! L_n^{\alpha}(x^2) \quad \text{and} \quad H_{2n+1}^{\alpha+\frac{1}{2}}(x) = (-1)^n 2^{2n+1} n! x L_n^{\alpha+1}(x^2),$$

where the  $L_n^{\alpha}$  are the Laguerre polynomials of index  $\alpha \geq -\frac{1}{2}$ , given by

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}).$$

For  $k = 1, 2, \dots$ , fix any positive integer  $n_0 > [\frac{k-1}{2}]$  and take for example the function defined on  $\mathbb{R}$  by  $\phi(x) = H_{2n_0}^{\alpha+\frac{1}{2}}(x) e^{-x^2}$ . Put  $P_i(x) = x^{2i}$  for  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$ , since  $P_i \in \text{span}_{\mathbb{R}}\{H_p^{\alpha+\frac{1}{2}}, p = 0, 1, \dots, 2[\frac{k-1}{2}]\}$ , then we can assert that  $\phi \in \mathcal{S}_*(\mathbb{R})$  and satisfy  $\int_0^{+\infty} x^{2i} \phi(x) d\mu_{\alpha}(x) = 0$ .

**Lemma 4.2** Let  $k = 1, 2, \dots$ ,  $1 < p < +\infty$  and  $\phi \in \mathcal{S}_*(\mathbb{R})$  such that  $\int_0^{+\infty} x^{2i} \phi(x) d\mu_{\alpha}(x) = 0$ , for all  $i \in \{0, 1, \dots, [\frac{k-1}{2}]\}$ , then there exists a constant  $c > 0$  such that for all  $f \in \mathcal{E}(\mathbb{R})$  satisfying  $\Lambda_{\alpha}^{2i} f \in L^p(\mu_{\alpha})$ ,  $0 \leq i \leq [\frac{k-1}{2}]$  and  $x > 0$ , we have

$$\|R_k(x, f) + R_k(-x, f)\|_{p,\alpha} \leq c \int_0^{+\infty} \min\left\{\left(\frac{x}{t}\right)^{k-1}, \left(\frac{x}{t}\right)^k\right\} \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t}. \quad (4.10)$$

**Proof.** Put for  $0 < \varepsilon < \delta < +\infty$

$$f_{\varepsilon,\delta}(y) = \int_{\varepsilon}^{\delta} (\phi_t *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t}, \quad y \in \mathbb{R}.$$

Then for  $i \in \mathbb{N}$ , we have

$$\Lambda_{\alpha}^{2i} f_{\varepsilon,\delta}(y) = \int_{\varepsilon}^{\delta} (\Lambda_{\alpha}^{2i} \phi_t *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t}, \quad y \in \mathbb{R}.$$

From the integral representation of  $\tau_x$ , we obtain by interchanging the orders of integration and (2.7),

$$\begin{aligned} \tau_x(f_{\varepsilon,\delta})(y) &= \int_{\varepsilon}^{\delta} \tau_x(\phi_t *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t} \\ &= \int_{\varepsilon}^{\delta} (\tau_x(\phi_t) *_{\alpha} \phi_t *_{\alpha} f)(y) \frac{dt}{t}, \quad y \in \mathbb{R}, \quad x \in (0, +\infty), \end{aligned}$$

so we can write for  $x \in (0, +\infty)$  and  $y \in \mathbb{R}$ ,

$$(R_k(x, f_{\varepsilon, \delta}) + R_k(-x, f_{\varepsilon, \delta}))(y) = \int_{\varepsilon}^{\delta} [(\tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t) *_{\alpha} \phi_t *_{\alpha} f](y) \frac{dt}{t}.$$

Using the Minkowski's inequality for integrals and (2.6), we get

$$\begin{aligned} & \| (R_k(x, f_{\varepsilon, \delta}) + R_k(-x, f_{\varepsilon, \delta})) \|_{p, \alpha} \\ & \leq \int_{\varepsilon}^{\delta} \| (\tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t) *_{\alpha} \phi_t *_{\alpha} f \|_{p, \alpha} \frac{dt}{t} \\ & \leq c \int_{\varepsilon}^{\delta} \| (\tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t) \|_{1, \alpha} \| \phi_t *_{\alpha} f \|_{p, \alpha} \frac{dt}{t} \\ & = c \int_{\varepsilon}^{\delta} \| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} \| \phi_t *_{\alpha} f \|_{p, \alpha} \frac{dt}{t}. \end{aligned} \quad (4.11)$$

For  $x, t \in (0, +\infty)$ , we have

$$\begin{aligned} & \| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} \\ & = \| \tau_x(\phi_t) + \tau_{-x}(\phi_t) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t \|_{1, \alpha} \\ & = \int_{\mathbb{R}} \left| \left( \int_{\mathbb{R}} \phi_t(z) (d\gamma_{x, y}(z) + d\gamma_{-x, y}(z)) \right) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}(x) \Lambda_{\alpha}^{2i} \phi_t(y) \right| d\mu_{\alpha}(y) \\ & = \int_{\mathbb{R}} \left| \left( \int_{\mathbb{R}} \phi\left(\frac{z}{t}\right) (d\gamma_{x, y}(z) + d\gamma_{-x, y}(z)) \right) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi\left(\frac{y}{t}\right) \right| t^{-2(\alpha+1)} d\mu_{\alpha}(y) \end{aligned} \quad (4.12)$$

By (2.2) and the change of variable  $z' = \frac{z}{t}$ , we have  $W_{\alpha}(x, y, z't) t^{2(\alpha+1)} = W_{\alpha}\left(\frac{x}{t}, \frac{y}{t}, z'\right)$ , which implies that  $d\gamma_{x, y}(z) = d\gamma_{\frac{x}{t}, \frac{y}{t}}(z')$ . Hence from (4.12), we obtain

$$\begin{aligned} & \| R_k(x, \phi_t) + R_k(-x, \phi_t) \|_{1, \alpha} \\ & = \int_{\mathbb{R}} \left| \left( \int_{\mathbb{R}} \phi(z') (d\gamma_{\frac{x}{t}, \frac{y}{t}}(z') + d\gamma_{\frac{-x}{t}, \frac{y}{t}}(z')) \right) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi\left(\frac{y}{t}\right) \right| t^{-2(\alpha+1)} d\mu_{\alpha}(y) \\ & = \int_{\mathbb{R}} \left| \left( \tau_{\frac{x}{t}}(\phi)\left(\frac{y}{t}\right) + \tau_{\frac{-x}{t}}(\phi)\left(\frac{y}{t}\right) \right) t^{-2(\alpha+1)} - 2 \left( \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi\right)_t(y) \right| d\mu_{\alpha}(y) \\ & = \left\| \left( \tau_{\frac{x}{t}}(\phi) + \tau_{\frac{-x}{t}}(\phi) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi \right) \right\|_{1, \alpha} \\ & = \left\| \tau_{\frac{x}{t}}(\phi) + \tau_{\frac{-x}{t}}(\phi) - 2 \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{2i}\left(\frac{x}{t}\right) \Lambda_{\alpha}^{2i} \phi \right\|_{1, \alpha} \\ & = \left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1, \alpha}. \end{aligned} \quad (4.13)$$

Since  $\phi \in \mathcal{S}_*(\mathbb{R})$ , then using (2.10) and (3.3), we can assert that

$$\left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^k \|\Lambda_\alpha^k \phi\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^k ,$$

on the other hand, by (3.4) we have

$$\left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^{k-1} \|\Lambda_\alpha^{k-1} \phi\|_{1,\alpha} \leq c \left(\frac{x}{t}\right)^{k-1} ,$$

then we get,

$$\left\| R_k\left(\frac{x}{t}, \phi\right) + R_k\left(\frac{-x}{t}, \phi\right) \right\|_{1,\alpha} \leq c \min \left\{ \left(\frac{x}{t}\right)^{k-1}, \left(\frac{x}{t}\right)^k \right\}. \quad (4.14)$$

From (4.11), (4.13) and (4.14), we obtain

$$\|(R_k(x, f_{\varepsilon,\delta}) + R_k(-x, f_{\varepsilon,\delta}))\|_{p,\alpha} \leq c \int_{\varepsilon}^{\delta} \min \left\{ \left(\frac{x}{t}\right)^{k-1}, \left(\frac{x}{t}\right)^k \right\} \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t}. \quad (4.15)$$

Note that  $\Lambda_\alpha^{2i} \phi *_{\alpha} \phi \in \mathcal{S}_*(\mathbb{R})$ . By (2.1) and (2.7), we have

$$\begin{aligned} \int_{\mathbb{R}} (\Lambda_\alpha^{2i} \phi *_{\alpha} \phi)(x) |x|^{2\alpha+1} dx &= 2^{\alpha+1} \Gamma(\alpha+1) \mathcal{F}_\alpha(\Lambda_\alpha^{2i} \phi *_{\alpha} \phi)(0) \\ &= 2^{\alpha+1} \Gamma(\alpha+1) \mathcal{F}_\alpha(\Lambda_\alpha^{2i} \phi)(0) \mathcal{F}_\alpha(\phi)(0) \\ &= 2^{\alpha+1} \Gamma(\alpha+1) \mathcal{F}_\alpha(\Lambda_\alpha^{2i} \phi)(0) \int_{\mathbb{R}} \phi(z) d\mu_\alpha(z) = 0. \end{aligned}$$

Since  $\Lambda_\alpha^{2i} \phi *_{\alpha} \phi$  is in the Schwartz space  $\mathcal{S}(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} |\log|x|| |\Lambda_\alpha^{2i} \phi *_{\alpha} \phi(x)| |x|^{2\alpha+1} dx < +\infty.$$

Then, by Calderón's reproducing formula related to the Dunkl operator (see [9], Theorem 3), we have

$$\lim_{\varepsilon \rightarrow 0, \delta \rightarrow +\infty} \Lambda_\alpha^{2i} f_{\varepsilon,\delta} = c \Lambda_\alpha^{2i} f, \quad \text{in } L^p(\mu_\alpha),$$

hence from (4.15), we deduce our result. ■

**Theorem 4.2** *Let  $0 < \beta < 1$ ,  $k = 1, 2, \dots$ ,  $1 < p < +\infty$  and  $1 \leq q \leq +\infty$ , then we have*

$$\widetilde{\mathcal{B}}^k \mathcal{D}_{\beta,\alpha}^{p,q} = \mathcal{C}_{p,q}^{k,\beta,\alpha},$$

*and for  $p = 1$ , we have only  $\widetilde{\mathcal{B}}^k \mathcal{D}_{\beta,\alpha}^{1,q} \subset \mathcal{C}_{1,q}^{k,\beta,\alpha}$ .*

**Proof.** Assume  $f \in \widetilde{\mathcal{B}}^k \mathcal{D}_{\beta,\alpha}^{p,q}$  for  $1 \leq p < +\infty$ ,  $1 \leq q \leq +\infty$  and  $r > \beta + k - 1$ .



- Case  $q = 1$ . By (4.5) and Fubini's theorem, we have

$$\begin{aligned}
\int_0^{+\infty} \frac{\|f *_{\alpha} \phi_t\|_{p,\alpha}}{t^{\beta+k-1}} \frac{dt}{t} &\leq c \int_0^{+\infty} \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{2(\alpha+1)}, \left( \frac{t}{x} \right)^r \right\} \tilde{\omega}_{p,\alpha}^k(x, f) t^{-\beta-k} dt \frac{dx}{x} \\
&\leq c \int_0^{+\infty} \tilde{\omega}_{p,\alpha}^k(x, f) \left( \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{2(\alpha+1)}, \left( \frac{t}{x} \right)^r \right\} t^{-\beta-k} dt \right) \frac{dx}{x} \\
&\leq c \int_0^{+\infty} \tilde{\omega}_{p,\alpha}^k(x, f) \left( x^{-r} \int_0^x t^{r-\beta-k} dt + x^{2(\alpha+1)} \int_x^{+\infty} t^{-\beta-k-2\alpha-2} dt \right) \frac{dx}{x} \\
&\leq c \int_0^{+\infty} \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \frac{dx}{x} < +\infty,
\end{aligned}$$

hence  $f \in \mathcal{C}_{p,1}^{k,\beta,\alpha}$ .

- Case  $q = +\infty$ . By (4.5), we have

$$\begin{aligned}
\|\phi_t *_{\alpha} f\|_{p,\alpha} &\leq c \left( \int_0^t \left( \frac{x}{t} \right)^{2(\alpha+1)} \tilde{\omega}_{p,\alpha}^k(x, f) \frac{dx}{x} + \int_t^{+\infty} \left( \frac{t}{x} \right)^r \tilde{\omega}_{p,\alpha}^k(x, f) \frac{dx}{x} \right) \\
&\leq c \sup_{x \in (0, +\infty)} \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \left( t^{-2(\alpha+1)} \int_0^t x^{2\alpha+\beta+k} dx + t^r \int_t^{+\infty} x^{\beta+k-r-2} dx \right) \\
&\leq c t^{\beta+k-1} \sup_{x \in (0, +\infty)} \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}},
\end{aligned}$$

then we deduce that  $f \in \mathcal{C}_{p,\infty}^{k,\beta,\alpha}$ .

- Case  $1 < q < +\infty$ . By (4.5) again, we have for  $t > 0$

$$\frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \leq c \int_0^{+\infty} \left( \frac{x}{t} \right)^{\beta+k-1} \min \left\{ \left( \frac{x}{t} \right)^{2(\alpha+1)}, \left( \frac{t}{x} \right)^r \right\} \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \frac{dx}{x}.$$

Put  $L(x, t) = \left( \frac{x}{t} \right)^{\beta+k-1} \min \left\{ \left( \frac{x}{t} \right)^{2(\alpha+1)}, \left( \frac{t}{x} \right)^r \right\}$  and  $q' = \frac{q}{q-1}$  the conjugate of  $q$ . Since

$$\int_0^{+\infty} L(x, t) \frac{dx}{x} = t^{-\beta-k-2\alpha-1} \int_0^t x^{\beta+k+2\alpha} dx + t^{-\beta-k+r+1} \int_t^{+\infty} x^{\beta+k-r-2} dx \leq c,$$

we can write using Hölder's inequality,

$$\begin{aligned}
\frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} &\leq c \int_0^{+\infty} (L(x, t))^{\frac{1}{q'}} \left( (L(x, t))^{\frac{1}{q}} \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right) \frac{dx}{x} \\
&\leq c \left( \int_0^{+\infty} L(x, t) \left( \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} \right)^{\frac{1}{q}}.
\end{aligned}$$

By the fact that

$$\int_0^{+\infty} L(x, t) \frac{dt}{t} = x^{\beta+k-r-1} \int_0^x t^{-\beta-k+r} dt + x^{\beta+k+2\alpha+1} \int_x^{+\infty} t^{-\beta-k-2\alpha-2} dt \leq c,$$

we get using Fubini's theorem,

$$\begin{aligned} \int_0^{+\infty} \left( \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} &\leq c \int_0^{+\infty} \left( \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \left( \int_0^{+\infty} L(x, t) \frac{dt}{t} \right) \frac{dx}{x} \\ &\leq c \int_0^{+\infty} \left( \frac{\tilde{\omega}_{p,\alpha}^k(x, f)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} < +\infty, \end{aligned}$$

which proves the result.

Assume now  $f \in \mathcal{C}_{p,q}^{k,\beta,\alpha}$  for  $1 < p < +\infty$  and  $1 \leq q \leq +\infty$ .

- Case  $q = 1$ . By (4.10) and Fubini's theorem, we have

$$\begin{aligned} \int_0^{+\infty} \frac{\tilde{\omega}_p^{\alpha}(f)(x)}{x^{\beta+k-1}} \frac{dx}{x} &\leq c \int_0^{+\infty} \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} \|\phi_t *_{\alpha} f\|_{p,\alpha} x^{-\beta-k} \frac{dt}{t} dx \\ &\leq c \int_0^{+\infty} \|\phi_t *_{\alpha} f\|_{p,\alpha} \left( \int_0^{+\infty} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} x^{-\beta-k} dx \right) \frac{dt}{t} \\ &\leq c \int_0^{+\infty} \|\phi_t *_{\alpha} f\|_{p,\alpha} \left( \frac{1}{t^k} \int_0^t x^{-\beta} dx + \frac{1}{t^{k-1}} \int_t^{+\infty} x^{-\beta-1} dx \right) \frac{dt}{t} \\ &\leq c \int_0^{+\infty} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \frac{dt}{t} < +\infty, \end{aligned}$$

then we obtain the result.

- Case  $q = +\infty$ . By (4.10), we get

$$\begin{aligned} \tilde{\omega}_p^{\alpha}(f)(x) &\leq c \left( \int_0^x \left( \frac{x}{t} \right)^{k-1} \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t} + \int_x^{+\infty} \left( \frac{x}{t} \right)^k \|\phi_t *_{\alpha} f\|_{p,\alpha} \frac{dt}{t} \right) \\ &\leq c \sup_{t \in (0, +\infty)} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \left( x^{k-1} \int_0^x t^{\beta-1} dt + x^k \int_x^{+\infty} t^{\beta-2} dt \right) \\ &\leq c x^{\beta+k-1} \sup_{t \in (0, +\infty)} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}}, \end{aligned}$$

so, we deduce that  $f \in \tilde{\mathcal{B}}^k \mathcal{D}_{\beta,\alpha}^{p,+\infty}$ .

- Case  $1 < q < +\infty$ . By (4.10) again, we have for  $x > 0$

$$\frac{\tilde{\omega}_p^{\alpha}(f)(x)}{x^{\beta+k-1}} \leq c \int_0^{+\infty} \left( \frac{t}{x} \right)^{\beta+k-1} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} \frac{\|\phi_t *_{\alpha} f\|_{p,\alpha}}{t^{\beta+k-1}} \frac{dt}{t}.$$

Note that

$$\left( \frac{t}{x} \right)^{\beta+k-1} \min \left\{ \left( \frac{x}{t} \right)^{k-1}, \left( \frac{x}{t} \right)^k \right\} = \left( \frac{t}{x} \right)^{\beta} \min \left\{ 1, \frac{x}{t} \right\}.$$

Put  $G(x, t) = \left( \frac{t}{x} \right)^{\beta} \min \left\{ 1, \frac{x}{t} \right\}$ . Since

$$\int_0^{+\infty} G(x, t) \frac{dt}{t} = x^{-\beta} \int_0^x t^{\beta-1} dt + x^{-\beta+1} \int_x^{+\infty} t^{\beta-2} dt \leq c,$$

then using Hölder's inequality, we can write

$$\begin{aligned} \frac{\tilde{\omega}_p^\alpha(f)(x)}{x^{\beta+k-1}} &\leq c \int_0^{+\infty} (G(x, t))^{\frac{1}{q'}} \left( (G(x, t))^{\frac{1}{q}} \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right) \frac{dt}{t} \\ &\leq c \left( \int_0^{+\infty} G(x, t) \left( \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}. \end{aligned}$$

By the fact that

$$\int_0^{+\infty} G(x, t) \frac{dx}{x} = t^{\beta-1} \int_0^t x^{-\beta} dx + t^\beta \int_t^{+\infty} x^{-\beta-1} dx \leq c,$$

we get using Fubini's theorem,

$$\begin{aligned} \int_0^{+\infty} \left( \frac{\tilde{\omega}_p^\alpha(f)(x)}{x^{\beta+k-1}} \right)^q \frac{dx}{x} &\leq c \int_0^{+\infty} \left( \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \left( \int_0^{+\infty} G(x, t) \frac{dx}{x} \right) \frac{dt}{t} \\ &\leq c \int_0^{+\infty} \left( \frac{\|\phi_t *_\alpha f\|_{p,\alpha}}{t^{\beta+k-1}} \right)^q \frac{dt}{t} < +\infty, \end{aligned}$$

thus the result is established. ■

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